Double integrals and changing the order of integration

Given the following double integral:

$$\int_{2}^{5}\int_{y-2}^{2y+1}(3xy+e^{x})\,dxdy$$

$$D=\{(x,y)\in\mathbb{R}^2/2\leq y\leq 5; y-2\leq x\leq 2y+1\}$$

a. Calculate the value of the double integral in the indicated area.
b. Graph the region. Change the order of integration and propose the integral.

Solution

1.

$$\int_{2}^{5}\int_{y-2}^{2y+1}(3xy+e^{x})\,dxdy$$

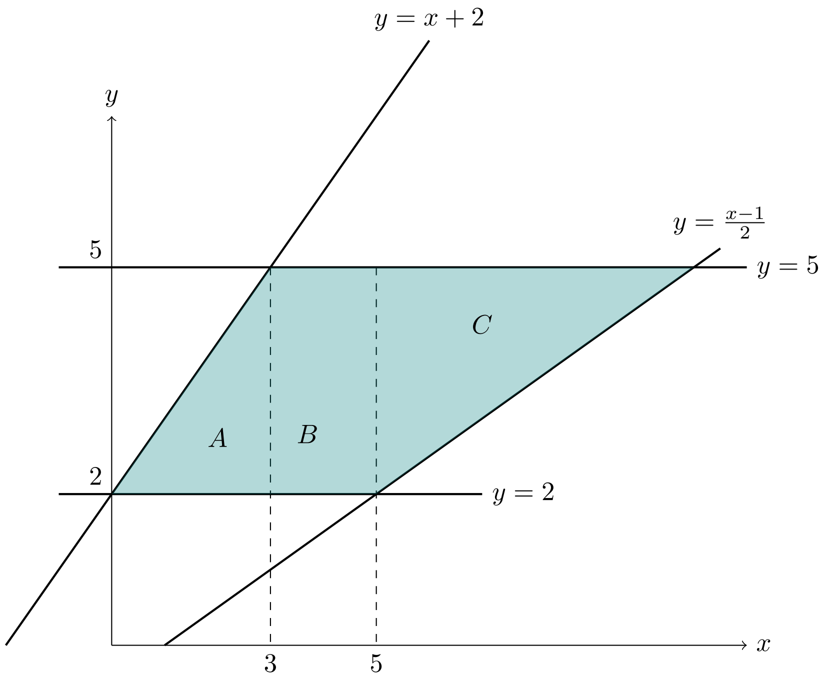
$$\int_{y-2}^{2y+1}(3xy+e^{x})\,dx = e^{2y+1}+2y-e^{y-2}+y+\frac{3y(y^{2}+8y-3)}{2}$$

$$=\int_{2}^{5}\left(e^{2y+1}+2y-e^{y-2}+y+\frac{3y(y^{2}+8y-3)}{2}\right)dy$$

$$=e^{2y+1}+2y-e^{y-2}+y+\frac{3y(y^{2}+8y-3)}{2}\Bigg|_{2}^{5}=\frac{8847}{8}+\frac{e^{13}-e^{7}-2e^{5}+2e^{2}}{2e^{2}}$$

$$=\frac{8847}{8}+\frac{e^{13}-e^{7}-2e^{5}+2e^{2}}{2e^{2}}=30949.65$$

2. Graph:



It is necessary to divide the area into 3 regions. In region A, regarding y, the floor is 2 and the ceiling is $x+2$:

$$\int_{0}^{3}\int_{2}^{x+2}(3xy+e^{x})\,dydx$$

$$\int_{2}^{x+2}(3xy+e^{x})\,dy = xe^{x}+3x\left(\frac{(2+x)^{2}}{2}-2\right)$$

$$=\int_{0}^{3}\left(xe^{x}+3x\left(\frac{(2+x)^{2}}{2}-2\right)\right)dx$$

$$\int_0^3 \left(xe^x + 3x\left(\frac{(2+x)^2}{2} - 2\right)\right) dx = 2e^3 + \frac{683}{8}$$

$$= 2e^3 + \frac{683}{8} \approx 125.55$$

In region B, the floor is 2 and the ceiling is 5.

$$B = \int_3^5 \int_2^5 (3xy + e^x)\, dydx$$

$$\int_2^5 (3xy + e^x)\, dy = 3e^x + \frac{63x}{2}$$

$$= \int_3^5 \left(3e^x + \frac{63x}{2}\right) dx$$

$$\int_3^5 \left(3e^x + \frac{63x}{2}\right) dx = 3\left(e^5 - e^3\right) + 252$$

$$= 3\left(e^5 - e^3\right) + 252 \approx 636.98$$

In region C, the floor is $x/2 - 1/2$ and the ceiling is 5:

$$\int_5^{11} \int_{\frac{x}{2}-\frac{1}{2}}^5 (3xy + e^x)\, dydx$$

$$\int_{\frac{x}{2}-\frac{1}{2}}^5 (3xy + e^x)\, dy = 11e^x - \frac{xe^x}{2} + 3x\left(\frac{25}{2} - \frac{(-1+x)^2}{8}\right)$$

$$= \int_5^{11} \left(11e^x - \frac{xe^x}{2} + 3x\left(\frac{25}{2} - \frac{(-1+x)^2}{8}\right)\right) dx$$

$$\int_5^{11} \left(11e^x - \frac{xe^x}{2} + 3x\left(\frac{25}{2} - \frac{(-1+x)^2}{8}\right)\right) dx = \frac{1539}{2} + \frac{e^{11} - 7e^5}{2}$$

$$= \frac{1539}{2} + \frac{e^{11} - 7e^5}{2} \approx 30187.12$$

And the sum of the results gives us the same value as before:

$$125.55 + 636.98 + 30187.12 = 30949.65$$